

ABSTRACT

Title of dissertation: \mathbb{Z}^d Symbolic Dynamics:
Coding with an Entropy Inequality

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In this paper we discuss subsystem and coding results in \mathbb{Z}^d symbolic dynamics for $d > 1$. We prove that any \mathbb{Z}^d shift of finite type with positive topological entropy has a family of subsystems of finite type whose entropies are dense in the interval from zero to the entropy of the original shift. We show a similar result for \mathbb{Z}^d sofic shifts, and also show every \mathbb{Z}^d sofic shift can be covered by a \mathbb{Z}^d shift of finite type arbitrarily close in entropy. We also show that if a \mathbb{Z}^2 shift of finite type with entropy greater than $\log N$ satisfies a certain mixing condition, then it must factor onto the full N -shift.

\mathbb{Z}^d Symbolic Dynamics: Coding with an Entropy Inequality

by

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DEDICATION

To my father, Vijay K. Desai

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Chapter 1

Introduction

1.1 Background

Discrete dynamical systems arose as a means to simplify the study of solutions of differential equations using cross sections to solutions. A discrete topological dynamical system is a phase space Y together with a continuous map $f : Y \rightarrow Y$. These systems can still be very difficult to understand. Symbolic dynamical systems, called shift spaces, arise when we make space discrete as well as time. The points in a shift space are infinite sequences (these can be bi-infinite, if f is a homeomorphism, or infinite only in one direction, otherwise), which represent the itineraries of points in the original dynamical system under f . Sometimes the system (Y, f) can be closely related to a shift space which is easier to study. In particular, one important type of shift space, the shifts of finite type, is strongly related to matrices, from which invariants can be extracted. The notion of shift spaces can be generalized to higher dimensions as well. In general, the points in a d -dimensional shift space are infinite d -dimensional arrays, which correspond to bi-infinite sequences for

$d = 1$.

Since this field's beginning, other uses for shift spaces have been found. For example, shift spaces give us schemes for data storage. In higher dimensions, shift spaces are closely related to the study of tiling spaces which appear in the study of quasi-crystals, percolation theory and statistical mechanics.

Natural coding questions arise in symbolic dynamics. For example, what are necessary and sufficient conditions for the existence of various types of codes between two shift spaces? More specifically, when is one a subsystem of the other? When are they 'the same'? When can one factor onto another? What are the invariants under conjugacy? We will discuss results addressing how two subshifts are related to one another, first discussing known results for $d = 1$, and then considering whether these results extend to higher dimensions.

1.2 Definitions and Notation

We begin by making the ideas mentioned above more precise. Let $\mathcal{A} = \{0, 1, \dots, N\}$, and let $X_{[N]} = \mathcal{A}^{\mathbb{Z}^d}$, $d \in \mathbb{N}$. Give \mathcal{A} the discrete topology, and then give $X_{[N]}$ the product topology. A point $x \in X_{[N]}$ can be viewed as an infinite d -dimensional array of symbols: for $\mathbf{w} \in \mathbb{Z}^d$, let $x_{\mathbf{w}}$ be the symbol in location \mathbf{w} .

For each $\mathbf{v} \in \mathbb{Z}^d$, define a shift map $\sigma_{\mathbf{v}} : x \mapsto y$ by $y_{\mathbf{w}} = x_{\mathbf{v}+\mathbf{w}}$, and let

σ be the \mathbb{Z}^d action $\{\sigma_{\mathbf{v}}\}_{\mathbf{v} \in \mathbb{Z}^d}$. The system $(X_{[N]}, \sigma)$ is the *full \mathbb{Z}^d N -shift*. For $R \subset \mathbb{Z}^d$, a *configuration* on R is some $\mathcal{M} \in \mathcal{A}^R$. For $x \in X_{[N]}$, denote the configuration occurring at R by x_R .

If X is a closed, shift invariant subset of $X_{[N]}$, then $(X, \sigma|_X)$ is called a \mathbb{Z}^d shift space, or subshift. Let \mathcal{A}_X be the symbol set of X . A configuration $\mathcal{M} \in \mathcal{A}^R$ is *allowed* in X if there is some $x \in X$ such that $x_R = \mathcal{M}$. Then we say that \mathcal{M} occurs in x .

As mentioned before, the most important subshifts are the shifts of finite type. A \mathbb{Z}^d subshift X is a *shift of finite type* (SFT) if it can be defined by forbidding a finite set of configurations $\mathcal{F} = \{F_1, F_2, \dots, F_m\}$ occurring in $(\mathcal{A}_X)^{\mathbb{Z}^d}$. X is a *one-step shift of finite type* if a point $x \in (\mathcal{A}_X)^{\mathbb{Z}^d}$ is allowed in X whenever $x_{\{\mathbf{m}, \mathbf{n}\}}$ is allowed for all $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$ with $\|\mathbf{m} - \mathbf{n}\| = 1$, where $\|\cdot\|$ is the Euclidean norm on \mathbb{R}^d . When $d = 1$, we may also define one-step SFTs as the set of bi-infinite walks along a directed graph. Every SFT may be recoded to be a one-step shift of finite type. We will assume that all shifts of finite type are one-step.

Let X, Y be subshifts. A map $\phi : X \rightarrow Y$ is a *block code* if it can be locally defined. That is, if $x \mapsto y$, then for all $\mathbf{v} \in \mathbb{Z}^d$, $y_{\mathbf{v}}$ depends on some finite block configuration occurring in x , centered at \mathbf{v} . The block codes from X to Y are exactly the continuous, shift-commuting maps. If ϕ is one-to-one,

then it is called an embedding. If ϕ is onto, it is called a factor code, or a factor map. If ϕ is both one-to-one and onto, then it is a conjugacy. A subshift X is a *sofic shift* if there exists a SFT Y and a factor code $\pi : Y \rightarrow X$.

Let $\Lambda(n) = \{\mathbf{v} = (v_1, \dots, v_d) \in \mathbb{Z}^d : 0 \leq v_i < n\}$. An n -block is a configuration on $\Lambda(n)$. Let $B_n(X)$ be the set of n -blocks allowed in X . Let $B(X) = \cup_n B_n(X)$.

The *topological entropy* of a d -dimensional subshift X is defined to be

$$h(X) = \lim_{n \rightarrow \infty} \frac{1}{n^d} \log |B_n(X)|$$

1.3 \mathbb{Z}^d subshifts for $d = 1$

In order to discuss \mathbb{Z} subshifts, we must introduce two definitions. A \mathbb{Z} subshift X is *irreducible* if given any two words $u, v \in B(X)$, there is an $r > 0$ and a word $w \in B_r(X)$ such that $uwv \in B(X)$. X is *topologically mixing* if for any $u, v \in B(X)$ there exists $r_0 > 0$ such that for any $r \geq r_0$, there exists $w \in B_r(X)$ such that $uwv \in B(X)$.

Entropy is one of the most useful tools in symbolic dynamics. If X is conjugate to Y , then $h(X) = h(Y)$. While entropy is not a complete invariant, it does tell us a great deal. If X and Y are SFTs and X embeds into Y , then it is easy to show that $h(X) \leq h(Y)$. If X embeds into Y , Y is an irreducible \mathbb{Z} SFT, and $h(X) = h(Y)$, then X and Y are conjugate. So to consider the

question of when a \mathbb{Z} subshift X embeds properly into an irreducible \mathbb{Z} SFT Y (that is, X embeds into Y but is not conjugate to Y), we only need to consider the case where $h(X) < h(Y)$. For X to embed into Y , there also needs to be a one-to-one shift commuting map from the periodic points of X into the periodic points of Y . We denote this condition $P(X) \hookrightarrow P(Y)$. Krieger proved that together these two conditions are also sufficient when Y is mixing.

Theorem 1.1. (Embedding Theorem) *[Kr 82] Let X be a \mathbb{Z} SFT, and let Y be a mixing \mathbb{Z} SFT. Then there is a proper embedding of X into Y if and only if $h(X) < h(Y)$ and $P(X) \hookrightarrow P(Y)$.*

We note that this theorem tells us that with some necessary restrictions on periodic points, a mixing SFT Y essentially has every SFT of less entropy as a subsystem.

Entropy is also closely tied to the existence of factor codes from X onto Y . If X factors onto Y , then we know that $h(X) \geq h(Y)$. There are counterexamples which show that if X factors onto Y and $h(X) = h(Y)$, then X and Y do not have to be conjugate. However, the existence of an entropy preserving factor code from X onto Y is still a strong condition.

Definition 1.2. *Let X be a SFT and let $\varphi : X \rightarrow Y$ be a factor code. A diamond for φ is a pair of distinct points $x, x' \in X$ such that $\varphi(x) = \varphi(x')$*

and such that $x_i = x'_i$ for all but finitely many $i \in \mathbb{Z}$.

Theorem 1.3. [LM 95] *Let X be an irreducible \mathbb{Z} SFT and $\varphi : X \rightarrow Y$ a factor code. Then the following are equivalent.*

(1) *φ is finite-to-one.*

(2) *φ has no diamonds.*

(3) *$h(X) = h(Y)$.*

In the case where X and Y are irreducible and $h(X) > h(Y)$, Boyle showed that there is a simple condition on periodic points that is equivalent to the existence of a factor code. There must be a shift-commuting map taking the periodic points of X into the periodic points of Y : i.e., if X has a periodic point of least period n , then Y must have a point whose least period divides n . This condition is denoted $P(X) \searrow P(Y)$.

Theorem 1.4. (Lower Entropy Factor Theorem) [Boy 83] *Let X and Y be irreducible \mathbb{Z} SFTs with $h(X) > h(Y)$. Then there is a factor map from X onto Y if and only if $P(X) \searrow P(Y)$*

We are also interested in the question of when shift spaces can be used to represent other dynamical systems.

Theorem 1.5. (Jewett-Krieger) *Let $T : X \rightarrow X$ be a finite entropy ergodic measure preserving transformation on a Lebesgue space (X, \mathcal{B}, μ) . Then T is*

measurably isomorphic to a strictly ergodic subshift with its invariant measure.

Taken together with Krieger's Embedding Theorem, this tells us that all traditional ergodic theory can be supported on the strictly ergodic subsystems of a mixing SFT, given necessary restrictions on entropy.

These results demonstrate the rich theory surrounding \mathbb{Z} SFTs and their subsystems. Most of them rely on the SFT being mixing. Since all mixing \mathbb{Z} SFTs have approximately the same structure of subsystems, studying \mathbb{Z} subshifts from this point is particularly effective. For \mathbb{Z} SFTs, various mixing conditions are equivalent. For \mathbb{Z}^d SFTs with $d > 1$, this is no longer true. Correspondingly, it turns out that for $d > 1$, mixing \mathbb{Z}^d SFTs are no longer homogeneous with respect to coding properties and richness of subsystems.

1.4 \mathbb{Z}^d subshifts for $d > 1$

For $d > 1$, difficulties arise immediately. A basic question is the following: given a finite set of forbidden configurations, can we determine if the SFT determined by this set is non-empty? This is called the non-emptiness problem. When $d = 1$, the answer is yes. In fact, it is well-known that a \mathbb{Z} SFT is non-empty if and only if it contains periodic points.

In 1961, Wang conjectured that there is a positive answer for $d > 1$ as well. Suppose we have a subshift whose alphabet is a set of square tiles

of equal size with variously colored edges (called Wang tiles). Two tiles can be placed next to each other when their edge colors match. Such a space is clearly a SFT, and in fact, any SFT will be conjugate to such a space. Since any non-empty \mathbb{Z} SFT has periodic points, it seems reasonable to approach the non-emptiness problem in terms of the existence of periodic points. Wang believed that given a set of Wang tiles, one can build out to larger and larger squares, and eventually come to either a square that cannot be completed, or a square that can tile the plane periodically.

Conjecture 1.6. (Wang) *[W 61] Any set of tiles that tiles the plane can be used to tile the plane periodically.*

The conjecture was proved false by Berger [Be 66], who found a set of (more than 20,000) tiles that can tile the plane aperiodically. Since that first example, many sets of tiles which only tile the plane aperiodically have been found. Thus Wang's initial idea - that the question of whether a SFT is nonempty is equivalent to the question of whether there are periodic points - does not work.

It turns out that without imposing other conditions on the subshifts under consideration, the answer to the first question is no; given a finite set of forbidden configurations, the non-emptiness problem is undecidable. By restricting their attention to specific classes of SFTs, Markley and Paul

have avoided these problems [MP 81],[MP2 81], as have Kitchens and Schmidt [KS 88],[KS 92]. We will not impose any such conditions on our SFTs, but we will assume all the SFTs we work with are non-empty.

As was previously discussed, in addition to topological mixing there are other mixing conditions in higher dimensions. The \mathbb{Z}^d definition of topologically mixing is the following. A shift space X is *topologically mixing* if for all finite subsets $R_1, R_2 \subset \mathbb{Z}^d$, there exists $l > 0$ such that for all $\mathbf{v} \in \mathbb{Z}^d$ with $d(R_1, R_2 + \mathbf{v}) > l$, and for all $x_1, x_2 \in X$ there exists a point $x \in X$ such that $x_{R_1} = (x_1)_{R_1}$ and $x_{R_2 + \mathbf{v}} = (x_2)_{R_2 + \mathbf{v}}$.

A SFT X has the *uniform filling property* (UFP) with filling length $l > 0$ if for all points $x_1, x_2 \in X$ and all rectangles $R \subset \mathbb{Z}^d$ there is a point $x \in X$ such that $x_R = (x_1)_R$ and $x_{B_l(R)^c} = (x_2)_{B_l(R)^c}$. Such subshifts are called *square mixing* by Lightwood.

A SFT X is *strongly irreducible* if there exists an $l \geq 0$ such that for all $x_1, x_2 \in X$ and all finite subsets $R_1, R_2 \subset \mathbb{Z}^d$ with $d(R_1, R_2) > l$, there is an $x \in X$ such that $x_{R_1} = (x_1)_{R_1}$ and $x_{R_2} = (x_2)_{R_2}$.

When $d = 1$, these three definitions all correspond to the definition of mixing given in the last section. For $d > 1$, a strongly irreducible SFT has the UFP, and a SFT with the UFP is topologically mixing. However, there are examples of SFTs which are topologically mixing, but do not have the UFP.

Currently there are no known examples of subshifts that have the UFP, but are not strongly irreducible, but we expect that such examples exist.

As was mentioned earlier, a \mathbb{Z} SFT X can be thought of as the edge shift associated with a directed graph G . If A is the adjacency matrix for G , then all the information about X is encoded in A . Many results about 1-dimensional SFTs are proved using matrix arguments on A . Most of these proofs do not extend to higher dimensions.

Many problems arise from these difficulties. For example, for $d = 1$ it is a simple matter to calculate the entropy of SFTs and sofic shifts using matrices. For $d > 1$, it is extremely difficult, if not impossible, to calculate entropy for most examples.

However, there are some partial coding results. When we restrict the class of subshifts we consider (most commonly by imposing a strong mixing condition), progress can be made. Rosenthal has proved a \mathbb{Z}^2 analogue of the Jewett-Krieger theorem.

Theorem 1.7. *[R 88] Let (X, \mathcal{B}, μ) be a Lebesgue space, and let $S : X \rightarrow X$ and $T : X \rightarrow X$ be commuting measure preserving transformations generating a free ergodic action on (X, \mathcal{B}, μ) . This \mathbb{Z}^2 action is measurably isomorphic to a strictly ergodic \mathbb{Z}^2 subshift with its invariant measure.*

We say that a dynamical system (Y, S) is a *universal model* if for every

aperiodic, ergodic, and measure preserving \mathbb{Z}^d action (X, μ, T) with $h(X, \mu, T) < h(Y, S)$, there exists an S -invariant borel probability measure ν such that (X, μ, T) and (Y, ν, S) are metrically isomorphic. Robinson and Sahin have results showing that a certain class of \mathbb{Z}^d SFTs are universal models for ergodic measure preserving \mathbb{Z}^d dynamical systems.

Theorem 1.8. *[RS 01] Let (Y, S) be a \mathbb{Z}^d SFT with the UFP and with dense periodic points. Then (Y, S) is a universal model.*

When $d = 2$, the UFP implies dense periodic points, so we only need to specify that the subshifts have the UFP.

We return to studying the way subsystems can relate to each other. There are other partial results. Lightwood has extended Krieger's embedding theorem for another class of \mathbb{Z}^d SFTs.

Theorem 1.9. *[L 03] For $d \geq 2$, let X be a non-periodic \mathbb{Z}^d subshift, let Z be a \mathbb{Z}^d finite-orbit SFT with the UFP, and suppose there is a homomorphism $X \rightarrow Z$. Then there exists an embedding $X \hookrightarrow Z$ if and only if $h(X) < h(Z)$.*

Lightwood then defines a stronger mixing condition for \mathbb{Z}^2 SFTs called square-filling-mixing. For this class of SFTs, he proves the following theorem.

Theorem 1.10. *[L 04] Let X be a non-periodic \mathbb{Z}^2 subshift and let Z be a \mathbb{Z}^2 square-filling-mixing SFT. There exists an embedding $X \hookrightarrow Z$ if and only if*

$$h(X) < h(Z).$$

In chapter 2, we will discuss the subsystems of a \mathbb{Z}^d sofic shift with positive entropy. When no further conditions are assumed, results are sparse. We will prove the following.

Theorem 1.11. *Let X be a SFT with $h(X) > 0$. Then there exists a family of SFT subsystems of X whose entropies are dense in $[0, h(X)]$.*

Proposition 1.12. *Let Y be a \mathbb{Z}^d sofic shift and $\varepsilon > 0$. Then Y has a SFT cover, $\pi : Z \rightarrow Y$ such that $h(Z) < h(Y) + \varepsilon$. Moreover the cover can be chosen so that for every subsystem Z' of Z , $h(Z') < h(\pi Z') + \varepsilon$.*

Theorem 1.13. *Let Y be a \mathbb{Z}^d sofic shift with $h(Y) > 0$. Then there exists a family of sofic subsystems whose entropies are dense in $[0, h(Y)]$.*

In chapter 3, we study the existence of factor maps from d -dimensional SFTs to the full N -shift. We discuss a recent result of Johnson and Madden for $d > 1$, and prove the following for $d = 2$.

Theorem 1.14. *Let X be a corner gluing \mathbb{Z}^2 SFT, and suppose $h(X) > \log N$. Then there exists a factor map $\varphi : X \rightarrow X_{[N]}$.*

Chapter 2

Subsystem entropy for \mathbb{Z}^d sofic systems

2.1 Introduction

As we discussed in chapter 1, a \mathbb{Z} SFT with positive entropy has a well understood wealth of subsystems. The structure of the subsystems of \mathbb{Z}^d SFTs for $d > 1$ is much less well understood. In the introduction, we gave some of the results known for more special classes of subshifts. But when we simply consider SFTs with positive entropy, very little is known.

Quas and Trow proved that for a \mathbb{Z}^d SFT X with positive entropy, $h(X)$ is an accumulation point for the entropies of the SFT subsystems of X .

Theorem 2.1. *[QT 00] Let X be a \mathbb{Z}^d SFT with $h(X) > 0$. Then for all $\varepsilon > 0$, there exists a proper subsystem Y of X , also of finite type, such that $h(X) - \varepsilon < h(Y) < h(X)$.*

We prove that for any \mathbb{Z}^d SFT X of positive entropy the SFT subsystems of X achieve dense entropies in $[0, h(X)]$. We show a similar result for sofic shifts: given a \mathbb{Z}^d sofic shift Y of positive entropy, the sofic subsystems of Y have entropies dense in $[0, h(Y)]$. We also construct a SFT cover with ε -close

entropy, answering an old question of Weiss (see Question 2.6).

2.2 Shifts of finite type

In this section, we show that if X is a \mathbb{Z}^d SFT with positive topological entropy, then there exist subsystems of X of finite type whose entropies are dense in $[0, h(X)]$. We begin by constructing a convenient cover for X . For simplicity, throughout this chapter, we will give arguments only for the case $d = 2$. The proofs for $d \neq 2$ are similar.

Construction 2.2. *Given a one-step \mathbb{Z}^d SFT X and $N \in \mathbb{N}$, we construct a certain \mathbb{Z}^d SFT $Z = Z_{(N)}$ and a shift-commuting surjection $\pi : Z \rightarrow X$.*

Without loss of generality, we may assume that X is a one-step SFT. We begin by defining the alphabet of Z , \mathcal{A}_Z , to be the disjoint union of two copies of the alphabet \mathcal{A}_X of X , where the letters from the first copy are colored red and the letters from the second copy are colored green. Let $\Pi : \mathcal{A}_Z \rightarrow \mathcal{A}_X$ be the map that forgets color. Define Z as follows: a point $x \in (\mathcal{A}_Z)^{\mathbb{Z}^2}$ will be in Z if and only if there is an $\mathbf{a} \in \mathbb{Z}^2$ such that $x_{\mathbf{n}}$ is red for all $\mathbf{n} \in M = \{\mathbf{a} + ((N\mathbb{Z} \times \mathbb{Z}) \cup (\mathbb{Z} \times N\mathbb{Z}))\}$, and $x_{\mathbf{n}}$ is green for all $\mathbf{n} \notin M$, and if removing color from x gives us a point in X . Thus, we may view a point in Z as a point in X with a gridlike color pattern imposed on it. Let Π define a one-block code $Z \rightarrow X$, denoted π . Then π is onto and everywhere $N^2 - to - 1$.

Remark 2.3. *Because π is finite-to-one, $h(Z') = h(\pi(Z'))$ for any subsystem Z' of Z . In particular, $h(Z) = h(X)$.*

Given any SFT X of positive entropy, and $\varepsilon > 0$, we use Construction 2.2 to find a family of subsystems of X with entropies ε -dense in $[0, h(X)]$. We then approximate those subsystems with SFTs which will also be contained in X . These SFTs will have entropy ε -dense in $[0, h(X)]$.

For $M \subset \mathbb{Z}^d$, define the border of M , denoted ∂M , to be the set of $\mathbf{v} \in M$ such that there exists \mathbf{w} in the complement of M such that $\|\mathbf{v} - \mathbf{w}\| = 1$.

Theorem 2.4. *Let X be a SFT with $h(X) > 0$. Then there exists a family of SFT subsystems of X whose entropies are dense in $[0, h(X)]$.*

Proof: Without loss of generality, we may assume that X is a one-step SFT. Let $\alpha = |\mathcal{A}_X|$, and let $\varepsilon > 0$. Choose $l \in \mathbb{N}$ such that $\log l - \log(l-1) < \varepsilon$. Since $h(X) > 0$, we can choose N large enough that at least one border of a block in $B_{N+1}(X)$ has more than l allowable interiors, and such that

$$(i) \frac{4N}{(N+1)^2} \log \alpha < \frac{\varepsilon}{3}, \quad (ii) \frac{1}{(N+1)^2} \log l < \frac{\varepsilon}{3}, \quad \text{and} \quad (iii) \frac{2}{(N+1)^2} \log N < \frac{\varepsilon}{3}.$$

For this N , construct $Z = Z_{(N)}$ and $\pi : Z \rightarrow X$ as in Construction 2.2. Let $B_{kN+1}^0(Z)$ be the set of $(kN+1)$ -blocks allowed in Z for which the border symbols are red.

We now construct in Z a finite decreasing sequence of SFTs, $\{Z_i\}_{i=0}^M$. Let $Z_0 = Z$, and define Z_i inductively from Z_{i-1} by disallowing one block of $B_{N+1}^0(Z_{i-1})$ whose (red) border has more than l allowable interiors. We end the sequence with Z_M , which is the last possible subsystem constructed in this manner. That is, for every $B \in B_{N+1}^0(Z_M)$, ∂B has at most l allowable interiors.

Claim: $h(Z_{i-1}) - h(Z_i) < \varepsilon$, for $i = 1, 2, \dots, M$

Every red border ∂B of some $B \in B_{N+1}^0(Z_{i-1})$ is also a border to some block in $B_{N+1}^0(Z_i)$. Thus the number of allowable interiors to ∂B in Z_i is at least $(\frac{l-1}{l}) \times$ the number of allowable interiors to ∂B in Z_{i-1} .

Given k , fix a choice of red symbols occurring in an element of $B_{kN+1}^0(Z_i)$.

Consider how many ways the block can be completed by filling in the green symbols:

$$\left| \begin{array}{l} \text{ways to complete block in } Z_i \\ \text{ways to complete block in } Z_{i-1} \end{array} \right| \geq \left(\frac{l-1}{l} \right)^{k^2} \left| \begin{array}{l} \text{ways to complete block in } Z_i \\ \text{ways to complete block in } Z_{i-1} \end{array} \right|.$$

Thus $|B_{kN+1}^0(Z_i)| \geq (\frac{l-1}{l})^{k^2} |B_{kN+1}^0(Z_{i-1})|$, and therefore

$$\begin{aligned}
h(Z_i) &= \lim_{k \rightarrow \infty} \left(\frac{1}{kN+1} \right)^2 \log |B_{kN+1}^0(Z_i)| \\
&\geq \lim_{k \rightarrow \infty} \left[\frac{1}{(kN+1)^2} \log \left(\frac{l-1}{l} \right)^{k^2} + \frac{1}{(kN+1)^2} \log |B_{kN+1}^0(Z_{i-1})| \right] \\
&= \frac{1}{N^2} \log \left(\frac{l-1}{l} \right) + h(Z_{i-1}) \\
&> h(Z_{i-1}) - \varepsilon
\end{aligned}$$

This proves the claim.

Now consider Z_M . There are at most α^{4N} red border configurations on blocks in $B_{N+1}^0(Z_M)$, and each border has at most l interiors, giving $|B_{N+1}(Z_M)| \leq \alpha^{4N} l N^2$. Thus

$$\begin{aligned}
h(Z_M) &\leq \frac{1}{(N+1)^2} \log |B_{N+1}(Z_M)| \\
&\leq \frac{1}{(N+1)^2} \log \alpha^{4N} + \frac{1}{(N+1)^2} \log l + \frac{1}{(N+1)^2} \log N^2 \\
&< \varepsilon
\end{aligned}$$

Consequently, the entropies $h(Z_M), \dots, h(Z_0)$ are ε -dense in $[0, h(Z)] = [0, h(X)]$.

Now the natural subsystems of X to consider are the subsystems $\{X_i\}_{i=0}^M$, where $X_i = \pi(Z_i)$. As noted earlier, $h(X_i) = h(Z_i)$. So there exist (sofic) subsystems of X whose entropies are ε -dense in $[0, h(X)]$. For any subsystem $X' \subset X$, there exists a decreasing sequence of shifts of finite type $\{R_i\}$ such

that $\bigcap_{i=1}^{\infty} R_i = X'$. Since X is a SFT, eventually the X_i will be contained in X . As the entropy function is upper semicontinuous on subshifts, $\lim_{i \rightarrow \infty} h(R_i) = h(X')$. Thus there are SFT subsystems of X whose entropies are ε -dense in $[0, h(X)]$. Let $\varepsilon \rightarrow 0$, and the proof is complete. \square

From the construction of the SFT subsystems of X , it is clear that we cannot guarantee them to be mixing. One might hope that the dense entropies could be achieved by SFTs with strong mixing properties. However, Quas and Sahin have constructed a topologically mixing \mathbb{Z}^2 SFT \overline{X} which shows that this is sometimes impossible.

Example [QS 03] There is a topologically mixing \mathbb{Z}^2 SFT \overline{X} and a number h_0 such that:

- $0 < h_0 < h(\overline{X})$
- If $Y \subset \overline{X}$ is a SFT which has the UFP, then $h(Y) \leq h_0$.
- For any ergodic σ -invariant weakly mixing measure μ , $h_\mu(\overline{X}) \leq h_0$

2.3 Sofic Shifts

In this section we turn our attention to \mathbb{Z}^d sofic shifts of positive entropy. We quote two old (though unpublished) questions of Weiss:

Question 2.5. *Does a \mathbb{Z}^d sofic shift have an equal entropy \mathbb{Z}^d SFT cover?*

Question 2.6. *Does a \mathbb{Z}^d sofic shift have a \mathbb{Z}^d SFT cover with ε -close entropy?*

(Smorodinsky also considered such questions around 1990).

Coven and Paul showed in [CP 75] that when $d = 1$, the answer to Question 2.5 is yes. For $d > 1$, the question is still open. We will answer Question 2.6 in the affirmative with the next proposition, using methods similar to those used in Construction 2.2.

Proposition 2.7. *Suppose Y is a \mathbb{Z}^d sofic shift and $\varepsilon > 0$. Then Y has a SFT cover, $\pi : Z \rightarrow Y$ such that $h(Z) < h(Y) + \varepsilon$. Moreover the cover can be chosen so that for every subsystem Z' of Z , $h(Z') < h(\pi Z') + \varepsilon$*

Proof: Since Y is sofic, there exists a SFT X and a continuous shift-commuting surjection $\phi : X \rightarrow Y$. Without loss of generality, we may assume that X and Y have disjoint alphabets, that X is a one-step SFT, and that ϕ is a 1-block code. Given $N \in \mathbb{N}$, construct a new SFT $Z = Z_{(N)}$ as follows: The alphabet of Z is the union of the alphabets of X and Y . For every allowable $(N + 1)$ -block in X , replace the interior symbols by their images under ϕ , and leave the border symbols as they are. Let $B_{N+1}^0(Z)$ be the set of such blocks. Construct a point in Z by laying these blocks together so they overlap on the corners. That is, $z \in (\mathcal{A}_Z)^{\mathbb{Z}^2}$ will be in Z if the following conditions hold: For some $\mathbf{a} \in \mathbb{Z}^2$, $z_{\mathbf{n}}$ is a symbol from \mathcal{A}_X for any $\mathbf{n} \in M = \mathbf{a} + ((N\mathbb{Z} \times \mathbb{Z}) \cup (\mathbb{Z} \times N\mathbb{Z}))$.

Otherwise, $z_{\mathbf{n}}$ is a symbol from \mathcal{A}_Y . Also, $z_{\mathbf{n}+\Lambda(N+1)} \in B_{N+1}^0(Z)$, for all $\mathbf{n} \in (\mathbf{a} + (N\mathbb{Z} \times N\mathbb{Z}))$.

Clearly, Z is a SFT. Define $\pi : Z \rightarrow Y$ by the one-block map which sends symbols from \mathcal{A}_X to their images under ϕ , and symbols from \mathcal{A}_Y to themselves. First, we observe that π is well-defined: For any block B in $B_{N+1}^0(Z)$, there exists a block B' in $B_{N+1}(X)$ with the same border, and such that when we replace the interior symbols of B' with their images under ϕ , we get B . For each $B \in B_{N+1}^0(Z)$, make a single choice of such an X -block. Since X is a 1-step shift of finite type, this defines a map $\psi : Z \rightarrow X$. For $z \in Z$, $\pi(z) = \phi(\psi(z)) \in Y$. Thus π is well-defined.

Next we observe that π is also onto: Let $y \in Y$. Since ϕ is onto, there exists $x \in X$ such that $\phi(x) = y$. Given this x , leave the symbols in $(\mathbb{Z} \times N\mathbb{Z}) \cup (N\mathbb{Z} \times \mathbb{Z})$ as they are, and replace all other symbols with their images under ϕ . This yields a point $z \in Z$ such that $\pi(z) = \phi(x) = y$.

We now show that when N is large, $h(Z) < h(Y) + \varepsilon$. Let $\alpha = |\mathcal{A}_X|$. There are at most α^{4N} possible borders for blocks in $B_{N+1}^0(Z)$. Therefore, $|B_{N+1}^0(Z)| \leq \alpha^{4N} |B_{N+1}(Y)|$. Thus we have the following inequalities:

$$\begin{aligned}
h(Z) &\leq \frac{1}{(N+1)^2} \log |B_{N+1}(Z)| \\
&\leq \frac{4N}{(N+1)^2} \log \alpha + \frac{1}{(N+1)^2} \log |B_{N+1}(Y)| \\
&< h(Y) + \varepsilon
\end{aligned}$$

for large enough N .

The final claim holds by a similar argument. \square

In the case where $d = 1$, it is well known that Theorem 2.4 is true if X is only assumed to be sofic. For $d > 1$ and X sofic, we are only able to show the existence of a family of sofic subsystems of X with dense entropies.

Theorem 2.8. *Let Y be a \mathbb{Z}^d sofic shift with $h(Y) > 0$. Then there exists a family of sofic subsystems whose entropies are dense in $[0, h(Y)]$.*

Proof: Let $\varepsilon > 0$. Construct the SFT cover Z of Y as in Proposition 2.7. Then by Theorem 2.4, there exists a family of SFT subsystems of Z with entropies dense in $[0, h(Z)]$. The images of these subsystems under π are sofic subsystems of Y with entropies ε -dense in $[0, h(Y)]$. Letting $\varepsilon \rightarrow 0$ completes the proof. \square

The next corollary follows immediately from this theorem.

Corollary 2.9. *Let Y be a \mathbb{Z}^d sofic shift, and let $a \in [0, h(Y)]$. Then there is a subsystem $X \subset Y$ such that $h(X) = a$.*

Proof: Using Theorem 2.8 recursively, we can find a decreasing sequence of subsystems of Y , X_i , whose entropies approach a . If we let $Y = \bigcap_{i=1}^{\infty} X_i$, then $h(Y) = a$. \square

Proposition 2.10. *Suppose $\phi : X \twoheadrightarrow Y$, X is a \mathbb{Z}^d SFT and Y is a \mathbb{Z}^d sofic shift. Given $\varepsilon > 0$ and a subsystem $Y' \subset Y$, there is a subsystem $X' \subset X$ such that $\phi : X' \twoheadrightarrow Y'$ and $h(X') < h(Y') + \varepsilon$. If Y' is sofic, then X' can be chosen to be sofic.*

Proof: Construct the SFT Z and $\pi : Z \twoheadrightarrow Y$ as in the proof of 2.7. Given a subsystem $Y' \subset Y$, let $Z' = \pi^{-1}(Y')$. Define $X' = \psi(Z')$, where $\psi : Z \rightarrow X$ is the function defined in the proof of Proposition 2.7. Then $\phi : X' \twoheadrightarrow Y'$, and since ψ is finite-to-one, $h(X') = h(Z')$. Then $h(X') < h(Y') + \varepsilon$. To prove the last statement of the Proposition, note that because Y' and Z are sofic, it follows that Z' is sofic, and therefore, X' is sofic. \square

Remark 2.11. *In the case $d = 1$, if Y' is sofic and transitive, then X' can be chosen sofic such that $\pi : X' \rightarrow Y'$ is bounded-to-1. This can easily be deduced from [MPW].*

Chapter 3

Corner gluing SFTs of sufficient entropy factor onto the full N -shift

3.1 Introduction

In this chapter, we prove that if a \mathbb{Z}^2 SFT X satisfies a mixing property called corner gluing and has entropy greater than $\log N$, then X factors onto the full N -shift. This answers a question posed by Johnson and Madden in [JM 05]. We will first give some background for this problem and discuss some previous results.

We consider the question of when a \mathbb{Z}^d SFT can factor onto the full N -shift. When $d = 1$, this question has been completely answered.

Theorem 3.1. *If X is a \mathbb{Z} SFT with $h(X) \geq \log n$, then X factors onto $X_{[n]}$.*

The proof of Theorem 3.1 for $h(X) > \log n$ relies on marker arguments that do not extend to $d > 1$. In the equal entropy case, the proof is an argument using eigenvectors and state splitting which relies very heavily on matrix-based state splitting for \mathbb{Z} SFTs, and as previously discussed, such matrix arguments also do not extend to higher dimensions.

As is often the case, the situation becomes much more difficult for $d > 1$, and there are only partial results. We would like to extend Theorem 3.1, but we must impose further conditions on X to get a similar result. One such condition is the *corner condition*. Let $\mathbf{c} = (1, 1, \dots, 1) \in \mathbb{Z}^d$. Define a *d-dimensional corner* to be $C = \{\mathbf{a} \in \{0, 1\}^d : \mathbf{a} \neq \mathbf{c}\}$ where \mathbf{c} is called the corner position.

Definition 3.2. *Let X be a SFT. X has corner condition N if, for every corner $C \in \mathcal{A}^C$, there are at least N allowable choices for the corner position \mathbf{c} .*

For $d \geq 1$, this is a sufficient condition for factoring onto the N -shift. In fact, when $d = 1$, $h(X) \geq \log N$ if and only if X is conjugate to an SFT satisfying corner condition N . However, when $d > 1$, while this condition certainly implies $h(X) \geq \log N$, the converse is not true.

In [JM 05], Johnson and Madden defined a new mixing condition called corner gluing. For $\mathbf{k} = (k_1, k_2, \dots, k_d) \in \mathbb{N}^d$, let $R_{\mathbf{k}} = \{(a_1, a_2, \dots, a_d) \in \mathbb{Z}^d : 0 \leq a_i < k_i \text{ for } 1 \leq i \leq d\}$.

Definition 3.3. *A \mathbb{Z}^d SFT X is corner gluing if there exists a gluing constant $g > 0$ such that given any two finite subsets $E_1, E_2 \subset \mathbb{Z}^2$ as defined below and allowable configurations \mathcal{C}_1 and \mathcal{C}_2 on these subsets, there exists a point*

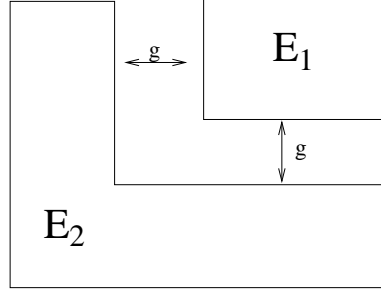


Figure 3.1: corner gluing

$x \in X$ with $x_{E_1} = \mathcal{C}_1$ and $x_{E_2} = \mathcal{C}_2$. $E_1 = R_{\mathbf{k}} + g\mathbf{c}$ for some $\mathbf{k} \in \mathbb{N}^d$ and $E_2 = (R_{\mathbf{k}'} - (\mathbf{k}' - \mathbf{k} - g\mathbf{c})) \setminus R_{\mathbf{k}+g\mathbf{c}}$ for some $\mathbf{k}' \in \mathbb{N}^d$ with $\mathbf{k}' > \mathbf{k} + \mathbf{c}$.

When we find such a rectangular configuration containing E_1 and E_2 , we say we are gluing E_1 to E_2 . We refer to the configurations needed to glue them together as gluing strips.

Using this mixing property, Johnson and Madden were able to make some progress in the case that the entropy of X is strictly greater than $\log N$ [JM 05].

Theorem 3.4. *Let X be a corner gluing \mathbb{Z}^d SFT with $h(X) > \log N$. Then X is the finite-to-one factor of a SFT \overline{X} that maps onto the full N -shift.*

They first note that if X is a SFT satisfying the above conditions, then for sufficiently large M , the higher power shift X^M will have corner condition N^{M^d} (with respect to the natural alphabet provided by M -blocks from X).

They then construct a SFT \overline{X} based on the M -blocks of X , and use the corner condition on X^M to show that \overline{X} factors onto the full N -shift.

They then posed the question of whether the extension \overline{X} is necessary. We prove that, at least for the case where $d = 2$, it is not. We expect this theorem will work for \mathbb{Z}^d SFTs with $d > 2$ with some modest elaboration of the proof.

Theorem 3.5. *Let X be a corner gluing \mathbb{Z}^2 SFT, and suppose $h(X) > \log N$. Then there exists a factor map $\varphi : X \rightarrow X_{[N]}$.*

3.2 Proof of Theorem 3.5

We begin the proof of Theorem 3.5 with a lemma that constructs a marker square M that is aperiodic for low periods. For $R \subset \mathbb{R}^d$ and $\mathbf{v} \in \mathbb{Z}^d \setminus \{0\}$, a configuration \mathcal{C} on R is said to be \mathbf{v} -periodic if for every pair $\mathbf{w}, \mathbf{w} + \mathbf{v} \in R \cap \mathbb{Z}^d$ we have $\mathcal{C}_{\mathbf{w}} = \mathcal{C}_{\mathbf{w}+\mathbf{v}}$.

Lemma 3.6. *Let X be a corner gluing \mathbb{Z}^2 SFT with $h(X) > 0$, and let g be the gluing constant. Then for $f, c \in \mathbb{N}$, if $F \in B_f(X)$, then there exists a square configuration $M \in B(X)$ as in figure 2, such that M is not \mathbf{v} -periodic whenever $\|\mathbf{v}\|_{\infty} < c$.*

Proof. Choose some $Q_0 \in B_c(X)$. Consider the $\mathbf{v} \in \mathbb{Z}^2$ such that $\|\mathbf{v}\|_{\infty} < c$

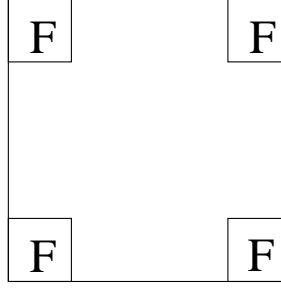


Figure 3.2: marker square M

and Q_0 is \mathbf{v} -periodic. Enumerate these as $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$.

Let $i \geq 1$. Let $l_0 = c$. Assume that $Q_{i-1} \in B_l(X)$, for some $l \in \mathbb{N}$ occurring with lower left corner at the origin, is not \mathbf{v}_j -periodic for $j \leq i-1$. Let $\mathbf{v}_i = (a, b)$. By symmetry, we may assume $a \geq 0$. The block Q_{i-1} will be the corner of the block Q_i , as pictured in Figure 3.3, according to the following cases: (i) $a, b > 0$, (ii) $a = 0, b > 0$, (iii) $a > 0, b = 0$, and (iv) $a > 0, b < 0$.

Consider case (i). Choose $k \in \mathbb{N}$ large enough that $ka, kb > l + g$ and suppose α is the symbol occurring at the lower left corner of Q_{i-1} . Since $h(X) > 0$, there is some $\beta \in \mathcal{A}_X$ with $\beta \neq \alpha$. Extend Q_{i-1} to an L -shape shown by the dashed lines in Figure 3.3(i), then glue the symbol β in at position $k\mathbf{v}_i$. Extend the resulting rectangle to a square Q_i . Q_i is not \mathbf{v}_i -periodic because \mathbf{v}_i -periodicity would imply $\alpha = \beta$. As Q_i has Q_{i-1} as a subblock, it is not \mathbf{v}_j -periodic for $j < i-1$ either. For the remaining three cases, though the picture changes, the argument remains the same (see Figure 3.3 (ii),(iii),(iv)).

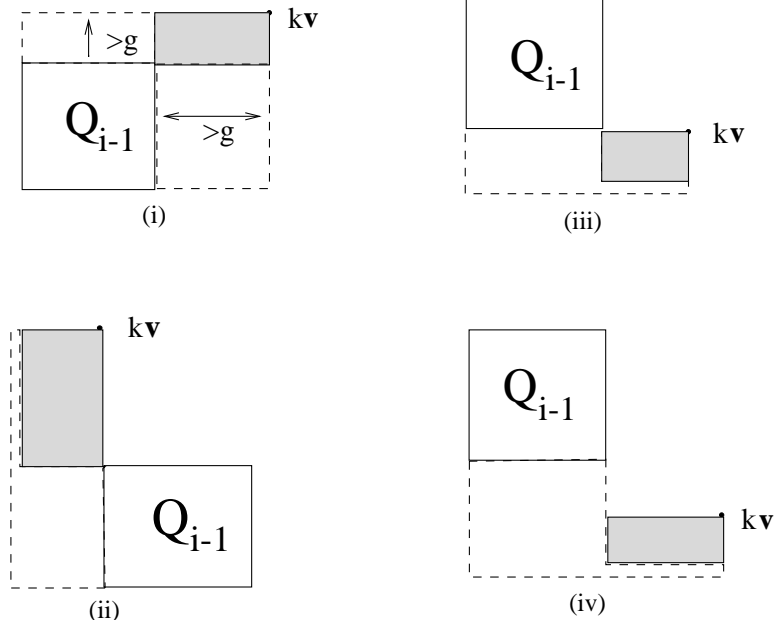


Figure 3.3: construction of Q_i

The construction of Q_i is the same, based on the corresponding figures. End this process with $Q = Q_p$. Then Q will not be \mathbf{v} -periodic for $\|\mathbf{v}\|_\infty < c$.

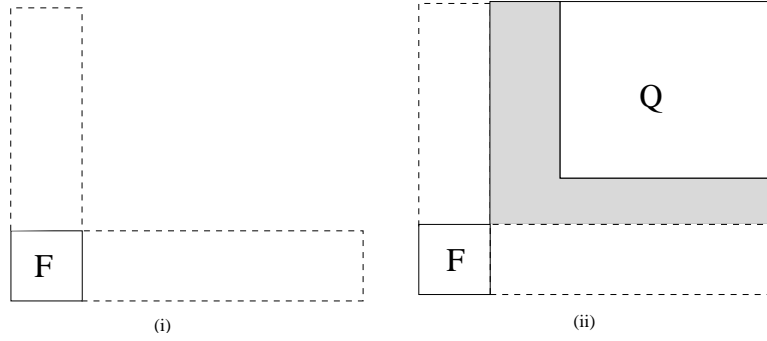


Figure 3.4: construction of M - step 1

Construct M beginning with F as follows. Extend F to an L -shaped configuration as in Figure 3.4(i). Then glue in Q as in Figure 3.4(ii), where the shaded region is the gluing region of width g necessary in the definition of corner gluing.

Extend this configuration to another L -shaped configuration, represented in Figure 3.5(i) with dashed lines. Choose some rectangle extension of F of the form seen in Figure 3.5(ii). Glue this rectangle to the L shaped configuration, to form a configuration as in Figure 3.5(iii).

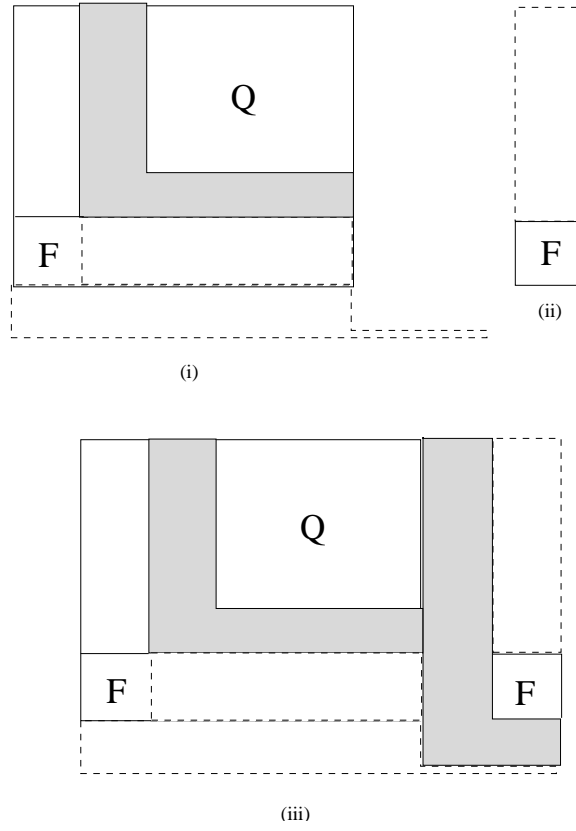


Figure 3.5: construction of M - step 2

Next, extend this rectangle to another L shape as in Figure 3.6(i), and choose some rectangle as in Figure 3.6(ii) with an F at the right and left ends. Glue these configurations together to form the square in Figure 3.6(iii). Take M to be the subblock with an F at each corner. \square

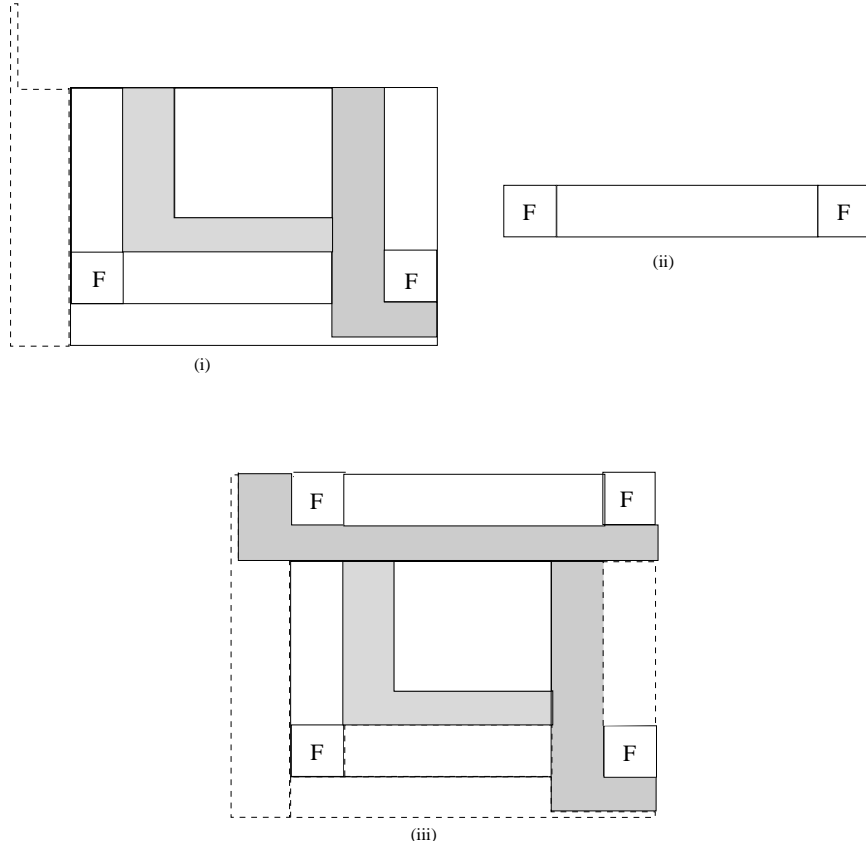


Figure 3.6: construction of M - step 3

With this lemma, we are ready to prove Theorem 3.5, using methods similar to those used by Johnson and Madden in their proof of Proposition 3.4.

Proof of Theorem 3.5. By Theorem 2.4, there is a proper subsystem Y of finite type in X with $h(Y) > \log N$. Choose some square configuration F that is forbidden in Y , and call its side length f . Construct a square configuration M using F as in Lemma 3.6 for $c = 2(f + g)$. Denote the side length of M by m .

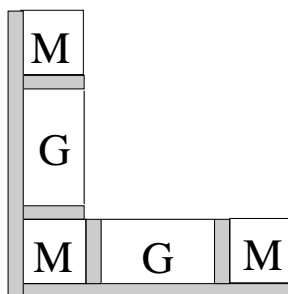


Figure 3.7: configuration \mathcal{L}

Now consider a configuration \mathcal{L} of the form seen in Figure 3.7, where the blocks labeled G can be filled in with any configuration of the appropriate size allowed in Y (we think of these as ‘good’ blocks), and the shaded regions are the necessary gluing strips. By the *inside corner* of \mathcal{L} , we mean the upper right corner of the block M in the lower left corner of \mathcal{L} .

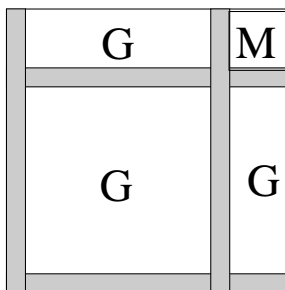


Figure 3.8: configuration \mathcal{C}

Glue a block \mathcal{C} of the type in Figure 3.8 to \mathcal{L} to get a block \mathcal{D} of the type in Figure 3.9. Such a block \mathcal{C} will be called a follower of \mathcal{L} . We do not

control the symbols in the gluing strips, but all G configurations of the correct size will appear in follower blocks for some choice of gluing strip configuration. Let l be the side length of the central block G in \mathcal{D} , and $J = l + 2g + m$; then $\mathcal{C} \in B_J(X)$. Each \mathcal{L} has at least $|B_l(Y)|$ followers and because $h(Y) > \log N$, we have $|B_l(Y)| > N^{J^2}$ for large enough l . For each \mathcal{L} , partition its followers into N^{J^2} nonempty sets, $P(\mathcal{L})_1, P(\mathcal{L})_2, \dots, P(\mathcal{L})_{N^{J^2}}$, depending only on the follower's central G block.

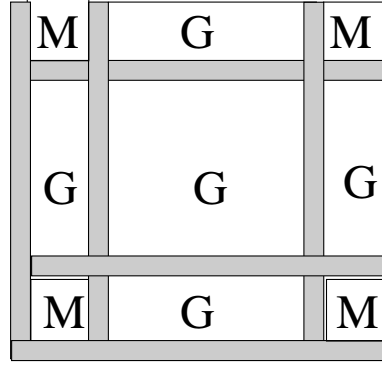


Figure 3.9: configuration \mathcal{D}

Claim: Let $x \in X$. If blocks \mathcal{D} and \mathcal{D}' of the form in 3.9 occur at different places in x , then their follower portions, \mathcal{C} and \mathcal{C}' , do not overlap.

Proof of claim. Without loss of generality, assume \mathcal{D} occurs with lower left corner at the origin, and \mathcal{D}' occurs with lower left corner at \mathbf{v} . Suppose \mathcal{C} and \mathcal{C}' do overlap. Recall that M is not \mathbf{v} -periodic whenever $\|\mathbf{v}\|_\infty < c$. Therefore, $\|\mathbf{v}\|_\infty \geq c$ because the lower left corner M 's cannot overlap too much. But

the lower left corner M of \mathcal{D}' cannot overlap too much with any other M in \mathcal{D} either, and we are assuming that \mathcal{C} and \mathcal{C}' overlap. Therefore $\|\mathbf{v}\|_\infty \leq J - c$. Now since M was constructed with an F at each corner and $c = 2(f + g)$, at least one subblock F of \mathcal{D}' must occur entirely in a ‘good’ block G of \mathcal{D} . However these blocks were chosen from the blocks allowed in Y , and so cannot contain F as a subblock. Thus \mathcal{C} and \mathcal{C}' cannot overlap. \square

Consider the J -blocks of $X_{[N]}$. Enumerate them as $E_1, E_2, \dots, E_{N^{J^2}}$. Now we are ready to construct a factor map $\varphi : X \rightarrow X_{[N]}$. We will define φ so that it essentially maps blocks from $P(\mathcal{L})_i$ to E_i for each configuration \mathcal{L} and $i = 1, 2, \dots, N^{J^2}$.

We make this precise as follows. For $x \in X$, suppose a configuration \mathcal{D} as in Figure 3.9 occurs in $x_{\Lambda(2J-1)+\mathbf{v}-J\mathbf{e}}$, the $2J - 1$ -block centered at \mathbf{v} , and $x_{\mathbf{v}}$ is in the follower portion of \mathcal{D} . By the claim, $x_{\mathbf{v}}$ occurs in the follower portion of no other such block \mathcal{D}' . Therefore, there exist unique $\mathbf{u}, \mathbf{w} \in \mathbb{Z}^2$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$, where \mathcal{L} has its inside corner at \mathbf{u} , and $0 < w_i \leq J$ for $i = 1, 2$. If $x_{\mathbf{v}}$ occurs in $\mathcal{C} \in P(\mathcal{L})_j$, then we define $\varphi(x)_{\mathbf{v}}$ to be the symbol from coordinate \mathbf{w} of E_j . If $x_{\mathbf{v}}$ is not in a follower, then $\varphi(x)_{\mathbf{v}} = 0$.

Claim: φ is onto.

Proof of claim. Let $E \in B_{kJ}(X_{[N]})$ be as in Figure 3.10. Choose a configuration \mathcal{R} of the form shown in Figure 3.11. Consider the configuration \mathcal{L} in the lower

If $E_{(0,0)} = E_i$, then choose a configuration $B_{(0,0)} \in P(\mathcal{L})_i$ as in Figure 3.12(i) to glue to \mathcal{L} . This new block $B_{(0,0)}$ together with \mathcal{R} forms two new \mathcal{L} configurations as shown in 3.12(ii). One will be above $B_{(0,0)}$ and one will be to the right of it. Glue in followers of each \mathcal{L} from the partition elements corresponding to $E_{(1,0)}$ and $E_{(0,1)}$. Continuing in this manner, complete a block $B \in B_{kJ}(X)$ as in Figure 3.12 that maps to E under the block map. \square

have corner condition 2 [JM 05]. Theorem 3.4 tells us only that X is the finite-to-one factor of a SFT that factors onto the full shift, and they ask whether X itself can factor onto $X_{[2]}$. Theorem 3.5 tells us that it does.

$$\mathbf{A}_h = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \quad \mathbf{A}_v = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

A number of open questions remain. The first, of course, is the question of whether Theorem 3.5 extends to $d > 2$. As stated above, we believe that it should. Then we would like to generalize Theorem 3.5 to SFTs X with entropy greater than $\log N$ which are not corner gluing. It is still unknown whether this is possible.

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